ON THE HOMOLOGY OF GRADED LIE ALGEBRAS

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Abstract. We consider the homology of finite dimensional graded Lie algebras with coefficients in a finite dimensional graded module. By a combinatorial approach we give a lower bound for their total homology. Our result extends a result of Deninger and Singhof for the case of trivial coefficients.

Applications for 2-step and free nilpotent Lie algebras are given.

1. Introduction

The (co)-homology of finite dimensional nilpotent Lie algebras $\mathfrak{g}$ is still not well understood. A major theme in the literature has been the construction of lower bounds for the Betti numbers $b_i(\mathfrak{g}) = \dim(\mathrm{H}^i(\mathfrak{g}, \mathbf{C}))$ (see [1], [4], [6]). Precise computations only exist in low dimensions and in particular cases (see [2], [3], [5], [9], [10]). General results are lacking and a number of conjectures are open.

In this paper we consider the homology of finite dimensional Lie algebras $\mathfrak{g}$ with coefficients in a finite dimensional graded $\mathfrak{g}$-module M. By a combinatorial approach we associate to $\mathfrak{g}$ and to M polynomials $p_{\mathfrak{g}}$ and q_M respectively and we prove that a lower bound for $\dim(\mathrm{H}_*(\mathfrak{g}, M))$ is given by the length of the product $p_{\mathfrak{g}} q_M$, i.e., the sum of the absolute values of the coefficients.

Deninger and Singhof [7] considered the homology of graded Lie algebras with trivial coefficients. It turns out that our method recovers their result.

Some applications for 2-step and free nilpotent Lie algebras are given.

All the Lie algebras and modules we will consider will be finite dimensional over a field $\mathbf{F}$ of characteristic zero. We use vertical bars $|\ |$ to denote dimension, i.e., $|A| = \dim A$.

2. A Weight Decomposition for $\mathrm{H}_*(\mathfrak{g}, M)$

If $\mathfrak{g}$ is a Lie algebra and M is a $\mathfrak{g}$-module, then the homology of $\mathfrak{g}$ with coefficients in M, $\mathrm{H}_*(\mathfrak{g}, M)$, is the homology of the Koszul complex $\left(\bigwedge \mathfrak{g} \otimes M, \partial\right)$; the differential ∂ is defined by:

$$\begin{aligned} (1)\quad \partial(x_1 \wedge \ldots \wedge x_p \otimes m) = &\sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \ldots \widehat{x_i} \wedge \ldots \widehat{x_j} \wedge \ldots x_p \otimes m \\ &+ \sum_{i=1}^{p} (-1)^i x_1 \wedge \ldots \widehat{x_i} \wedge \ldots x_p \otimes x_i m. \end{aligned}$$

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We introduce the following general notation: Let $W = \bigoplus_{j=1}^N W_j$ and $M = \bigoplus_{i=1}^m M_i$ be two graded vector spaces. For each N-tuple $(j_1, \dots, j_N)$ of non-negative integers and each $1 \le i \le m$ consider the subspace

$$V_{j_1,\dots,j_N,i} = \bigwedge^{j_1} W_1 \otimes \cdots \otimes \bigwedge^{j_N} W_N \otimes M_i$$

of $\bigwedge W \otimes M$ and declare it to be a subspace of weight

$$\omega(j_1, \dots, j_N, i) = j_1 + 2j_2 + \cdots + Nj_N + i.$$

Furthermore, given a positive integer α, let $\bigwedge W \otimes M(\alpha)$ be the sum of the subspaces of weight α. We call α a weight for (W, M) if $\bigwedge W \otimes M(\alpha) \neq 0$.

It is clear that we have a weight decomposition

$$\bigwedge W \otimes M = \bigoplus_\alpha \bigwedge W \otimes M(\alpha).$$

In particular, let $\mathfrak{g} = \bigoplus_{j=1}^N \mathfrak{a}_i$ be a graded Lie algebra and $M = \bigoplus_{i=1}^m M_i$ a graded $\mathfrak{g}$-module. That is

$$[\mathfrak{a}_i, \mathfrak{a}_j] \subseteq \mathfrak{a}_{i+j} \quad \text{and} \quad \mathfrak{a}_j M_i \subset M_{i+j}. \tag{2}$$

Then we have a weight decomposition

$$\bigwedge \mathfrak{g} \otimes M = \bigoplus_\alpha \bigwedge \mathfrak{g} \otimes M(\alpha)$$

of the Koszul complex.

Proposition 2.1. *The differential ∂ preserves the weight decomposition of $\bigwedge \mathfrak{g} \otimes M$, inducing a weight decomposition of the homology of $\mathfrak{g}$ with coefficients in M;*

$$\mathrm{H}_*(\mathfrak{g}, M) = \bigoplus_\alpha \mathrm{H}_*(\mathfrak{g}, M)(\alpha),$$

where $\mathrm{H}_*(\mathfrak{g}, M)(\alpha)$ is the homology of the subcomplex $\big(\bigwedge \mathfrak{g} \otimes M(\alpha), \partial\big)$.

Proof. Let α be a weight for $(\mathfrak{g}, M)$ and take $x \in \bigwedge \mathfrak{g} \otimes M(\alpha)$. According to (1) consider separately the two summands of $\partial(x)$; the first one is in $\bigwedge \mathfrak{g} \otimes M(\alpha)$ because $\mathfrak{g}$ is graded and the second summand is also in $\bigwedge \mathfrak{g} \otimes M(\alpha)$ because M is graded (see (2)). $\square$

3. A Combinatorial Formula

For each graded vector space $W = \bigoplus_{j=1}^N W_j$ consider the subspaces $W_{\text{even}} = \bigoplus_j W_{2j}$ and $W_{\text{odd}} = \bigoplus_j W_{2j+1}$.

Let $\big(\mathcal{C} = (C_i), \partial\big)$ be a finite complex of finite dimensional vector spaces, then it is well known that

$$\big|\mathrm{H}_*(\mathcal{C})\big| \ge \big||\mathcal{C}_{\text{even}}| - |\mathcal{C}_{\text{odd}}|\big|.$$

We will compute this lower bound for each one of the subcomplexes $\big(\bigwedge \mathfrak{g} \otimes M(\alpha), \partial\big)$ of Proposition 2.1.

Consider the subspaces $g_k = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$ of $\mathfrak{g}$. For each integer α and each $1 \le k \le N$, let

$$h_k(\alpha) = \begin{cases} \big|(\bigwedge g_k \otimes M(\alpha))_{\text{even}}\big| - \big|(\bigwedge g_k \otimes M(\alpha))_{\text{odd}}\big|, & \text{if } \alpha \text{ is a weight for } (g_k, M); \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.1. *Given $\alpha \geq 1$ and $2 \leq k \leq N$, the number $h_k(\alpha)$ can be recursively computed by:*

$$h_k(\alpha) = \sum_{i_k=0}^{|\mathfrak{a}_k|} (-1)^{i_k} h_{k-1}(\alpha - i_k k) \binom{|\mathfrak{a}_k|}{i_k}.$$

Proof. We first notice that $\left(\bigwedge g_k \otimes M(\alpha)\right)_{\text{even/odd}} = \left(\bigwedge g_k\right)_{\text{even/odd}} \otimes M(\alpha)$. By decomposing $g_k = g_{k-1} \oplus \mathfrak{a}_k$ we get the following decompositions for the even and odd subcomplex.

$$\begin{aligned}
\left(\bigwedge g_k\right)_{\text{even}} \otimes M(\alpha) &= \left(\bigwedge g_{k-1}\right)_{\text{even}} \otimes M(\alpha) \oplus \left(\bigwedge g_{k-1}\right)_{\text{odd}} \otimes \mathfrak{a}_k \otimes M(\alpha) \\
&\quad \oplus \left(\bigwedge g_{k-1}\right)_{\text{even}} \otimes \bigwedge^2 \mathfrak{a}_k \otimes M(\alpha) \oplus \dots \\
\left(\bigwedge g_k\right)_{\text{odd}} \otimes M(\alpha) &= \left(\bigwedge g_{k-1}\right)_{\text{odd}} \otimes M(\alpha) \oplus \left(\bigwedge g_{k-1}\right)_{\text{even}} \otimes \mathfrak{a}_k \otimes M(\alpha) \\
&\quad \oplus \left(\bigwedge g_{k-1}\right)_{\text{odd}} \otimes \bigwedge^2 \mathfrak{a}_k \otimes M(\alpha) \oplus \dots
\end{aligned}$$

From these decompositions we get the following expression for $h_k(\alpha)$:

$$\begin{aligned}
(3) \quad h_k(\alpha) &= \left|\left(\bigwedge g_k\right)_{\text{even}} \otimes M(\alpha)\right| - \left|\left(\bigwedge g_k\right)_{\text{odd}} \otimes M(\alpha)\right| \\
&= \sum_{i_k=0}^{|\mathfrak{a}_k|} (-1)^{i_k} \left|\left(\bigwedge g_{k-1}\right)_{\text{even}} \otimes \bigwedge^{i_k} \mathfrak{a}_k \otimes M(\alpha)\right| - \left|\left(\bigwedge g_{k-1}\right)_{\text{odd}} \otimes \bigwedge^{i_k} \mathfrak{a}_k \otimes M(\alpha)\right|
\end{aligned}$$

It is clear that $\left|V_{j_1,\dots,j_k,\dots,j_N,i} \otimes \bigwedge^{i_k} \mathfrak{a}_k\right| = \left|V_{j_1,\dots,j_k+i_k,\dots,j_N,i}\right|$; on the other hand if $j_1 + 2j_2 + \dots + k(j_k + i_k) + \dots + Nj_n + i = \alpha$, then $j_1 + 2j_2 + \dots + Nj_n + i = \alpha - i_k k$. Therefore,

$$\left|\left(\bigwedge g_{k-1}\right)_{\text{even/odd}} \otimes \bigwedge^{i_k} \mathfrak{a}_k \otimes M(\alpha)\right| = \left|\left(\bigwedge g_{k-1}\right)_{\text{even/odd}} \otimes M(\alpha - i_k k)\right| \binom{|\mathfrak{a}_k|}{i_k}.$$

The Lemma follows by replacing the summands of (3) according to this identity. □

Proposition 3.2. *Let $\mathfrak{g} = \oplus_{j=1}^N \mathfrak{a}_j$ be a graded Lie algebra and let $M = \oplus_{i=1}^m M_i$ be a graded $\mathfrak{g}$-module. Then for any $\alpha \geq 1$,*

$$(4) \quad h_N(\alpha) = \sum_{i=1}^{m} (-1)^{\alpha - i} \left(\sum_{i_N=0}^{|\mathfrak{a}_N|} \cdots \sum_{i_2=0}^{|\mathfrak{a}_2|} (-1)^{i_2 + 2i_3 + \dots + (N-1)i_N} \binom{|\mathfrak{a}_N|}{i_N} \cdots \binom{|\mathfrak{a}_2|}{i_2} \binom{|\mathfrak{a}_1|}{\alpha - Ni_N - \dots - 2i_2 - i} \right) |M_i|.$$

Proof. By applying recursively Lemma 3.1 we get

$$h_N(\alpha) = \sum_{i_N=0}^{|\mathfrak{a}_N|} \cdots \sum_{i_2=0}^{|\mathfrak{a}_2|} (-1)^{i_N + \dots + i_2} \binom{|\mathfrak{a}_N|}{i_N} \cdots \binom{|\mathfrak{a}_2|}{i_2} h_1(\alpha - Ni_N - \dots - 2i_2).$$

One can easily verify that for any $\beta \geq 1$

$$h_1(\beta) = \sum_{i=1}^{m} (-1)^{\beta-i} \binom{|\mathfrak{a}_1|}{\beta - i} |M_i|.$$

The proposition follows by putting together these two formulas. $\square$

Remark 3.1. If $M = \mathbf{F}$ is the trivial $\mathfrak{g}$-module, then the recursive formula in Lemma 3.1 is still valid; on the other hand $h_1(\beta) = (-1)^\beta \binom{|\mathfrak{a}_1|}{\beta}$ and thus

$$h_N(\alpha) = (-1)^\alpha \sum_{i_N,\dots,i_2} (-1)^{i_2+2i_3+\dots+(N-1)i_N} \binom{|\mathfrak{a}_N|}{i_N} \dots \binom{|\mathfrak{a}_2|}{i_2} \binom{|\mathfrak{a}_1|}{\alpha - Ni_N - \dots - 2i_2}.$$

4. Integral Representation of Combinatorial Sums

We come now to the evaluation of the combinatorial formula for $h_N(\alpha)$ in Proposition 3.2. To do this we follow the method described in [8].

The starting point is the following basic identity: the integral representation of the binomial coefficients,

$$\binom{n}{k} = \frac{1}{2\pi i} \oint \frac{(1+w)^n}{w^{k+1}} dw.$$

Example. Let us compute $S_2(\alpha) = \sum_{k=0}^{r_2} (-1)^k \binom{r_2}{k} \binom{r_1}{\alpha - 2k}$, for fixed r_1 and r_2.

$$\begin{aligned}
S_2(\alpha) &= \frac{1}{(2\pi i)^2} \sum_{k=0}^{\infty} (-1)^k \oint \frac{(1+w)^{r_2}}{w^{k+1}} dw \oint \frac{(1+v)^{r_1}}{v^{\alpha-2k+1}} dv, \\
&= \frac{1}{(2\pi i)^2} \sum_{k=0}^{\infty} (-1)^k \oint \frac{(1+w)^{r_2}(1+v)^{r_1} v^{2k}}{w v^{\alpha+1} w^k} dw dv, \\
&= \frac{1}{(2\pi i)^2} \int_{\substack{|w|=2 \\ |v|=1}} \frac{(1+w)^{r_2}(1+v)^{r_1}}{w v^{\alpha+1}} \left(\sum_{k=0}^{\infty} \left(\frac{-v^2}{w} \right)^k \right) dw dv, \\
&= \frac{1}{(2\pi i)^2} \int_{\substack{|w|=2 \\ |v|=1}} \frac{(1+w)^{r_2}(1+v)^{r_1}}{v^{\alpha+1}(w+v^2)} dw dv.
\end{aligned}$$

By the residue theorem $\int_{|w|=2} \frac{(1+w)^{r_2}}{w+v^2} = 2\pi i (1-v^2)^{r_2}$. Now it is clear that

$$S_2(\alpha) = \text{degree } \alpha \text{ coefficient of } p_{(r_1,r_2)}(x),$$

where $p_{(r_1,r_2)}(x) = (1+x)^{r_1}(1-x^2)^{r_2}$.

Lemma 4.1. *Let $r_1, \dots, r_k$ be non-negative integers and let* $S_k(\alpha) = \sum_{i_k,\dots,i_2} (-1)^{i_2+2i_3+\dots+(k-1)i_k} \binom{r_k}{i_k} \dots \binom{r_2}{i_2} \binom{r_1}{\alpha - ki_k - \dots - 2i_2}$. *Then,*

$$S_k(\alpha) = \text{degree } \alpha \text{ coefficient of } p_{(r_1,\dots,r_k)}(x),$$

where $p_{(r_1,\dots,r_k)}(x) = \prod_{i=1}^{k} \left(1 + (-1)^{i+1} x^i\right)^{r_i}$.

Proof. The case $k = 2$ has been proved in the previous Example, thus we proceed by induction on k. Write

$$S_k(\alpha) = \sum_{i_k=0}^{r_k} (-1)^{(k-1)i_k} \binom{r_k}{i_k} \sum_{i_{k-1},\dots,i_2} (-1)^{i_2+2i_3+\dots+(k-2)i_{k-1}} S_{k-1}(\alpha - ki_k).$$

Since $S_{k-1}(\alpha - ki_k) = \frac{1}{2\pi i}\oint \frac{p_{(r_1,\ldots,r_{k-1})}(v)}{v^{\alpha-ki_k+1}}dv$, hence

$$\begin{aligned} S_k(\alpha) &= \frac{1}{(2\pi i)^2}\sum_{i_k=0}^{\infty}(-1)^{(k-1)i_k}\oint \frac{(1+w)^{r_k}}{w^{i_k+1}}\frac{p_{(r_1,\ldots,r_{k-1})}(v)}{v^{\alpha-ki_k+1}}dwdv \\ &= \frac{1}{(2\pi i)^2}\int_{\substack{|w|=2\\|v|=1}}\frac{p_{(r_1,\ldots,r_{k-1})}(v)(1+w)^{r_k}}{v^{\alpha+1}w}\sum_{i_k=0}^{\infty}\left(\frac{(-1)^{k-1}v^k}{w}\right)^{i_k}dwdv. \end{aligned}$$

The geometric series inside the integral is equal to $\frac{w}{w+(-1)^k v^k}$ and by the residue theorem $\int_{|w|=2}\frac{(1+w)^{r_k}}{w+(-1)^k v^k} = 2\pi i(1+(-1)^{k+1}v^k)^{r_k}$. Finally, since $p_{(r_1,\ldots,r_{N-1})}(v)\big(1+(-1)^{N+1}v^N\big)^{r_N} = p_{(r_1,\ldots,r_N)}(v)$ the lemma is proved. □

Remark 4.1. It turns out that $p_{(|\mathfrak{a}_1|,\ldots,|\mathfrak{a}_N|)}(-x)$ is the polynomial associated by Deninger and Singhof to $\mathfrak{g}(=\oplus_{j=1}^N\mathfrak{a}_j)$ (see [7]).

5. The Main Results

Definition. If P is a polynomial, the *length* of P, $L(P)$, is the sum of the absolute values of all the coefficients of P.

Let $\mathfrak{g}$ be a graded Lie algebra and M a graded $\mathfrak{g}$-module. If $\mathfrak{g} = \oplus_{j=1}^N\mathfrak{a}_j$ and $M = \oplus_{i=1}^m M_i$, then we associate to $\mathfrak{g}$ and to M the following polynomials:

$$\begin{aligned} p_{\mathfrak{g}}(x) &= \prod_{j=1}^{N}(1-x^j)^{r_j}, \text{ where } r_j = |\mathfrak{a}_j|; \\ q_M(x) &= \sum_{i=1}^{m}|M_i|x^{i-1}. \end{aligned}$$

We will refer to $p_{\mathfrak{g}}$ and to q_M as the associated polynomials to $\mathfrak{g}$ and to M respectively.

Theorem 5.1. *Let $\mathfrak{g}$ be a graded Lie algebra and let M be a graded $\mathfrak{g}$-module. Let $p_{\mathfrak{g}}$ and q_M be their associated polynomials. Then,*

$$\big|\mathrm{H}_*(\mathfrak{g}, M)\big| \geq L(p_{\mathfrak{g}}q_M).$$

Proof. According to Proposition 3.2 $h_N(\alpha) = \sum_{i=1}^m(-1)^{\alpha-1}S_N(\alpha)|M_i|$, where $r_j = |\mathfrak{a}_j|$ in $S_N(\alpha)$. Since $S_N(\alpha)$ is the α-coefficient of $p_{\mathfrak{g}}(-x)$ (see Lemma 4.1 and the definition of $p_{\mathfrak{g}}$), then $h_N(\alpha)$ is the $(\alpha-1)$-coefficient of the product $p_{\mathfrak{g}}(-x)q_M(-x)$.

From Proposition 2.1 and the definition of $h_N(\alpha)$ it follows that

$$\big|\mathrm{H}_*(\mathfrak{g}, M)\big| = \sum_{\alpha}\big|\mathrm{H}_*(\mathfrak{g}, M)(\alpha)\big| \geq \sum_{\alpha}\big|h_N(\alpha)\big| = L(p_{\mathfrak{g}}(-x)q_M(-x)).$$

Since $L(P(x)) = L(P(-x))$ for any polynomial P the proof is completed. □

Theorem 5.2. *Let $\mathfrak{g} = \oplus_{j=1}^N\mathfrak{a}_j$ be a graded Lie algebra and let M be a graded $\mathfrak{g}$-module. Let q_M be the associated polynomial to M and let $\frac{N}{2} < k \leq N$ be a fixed integer. If $q_M(e^{\frac{\pi i}{k}}) \neq 0$, then*

$$\big|\mathrm{H}_*(\mathfrak{g}, M)\big| \geq 2^{|\mathfrak{a}_k|}.$$

Proof. In [7] (Proposition 4.3) it has been proved that $L(p_{\mathfrak{g}}) \geq 2^{|\mathfrak{a}_k|}$. We mimic that proof to show that $L(p_{\mathfrak{g}} q_M) \geq 2^{|\mathfrak{a}_k|}$.

Consider the algebra $\mathrm{M}_r(\mathbf{C})$ of $r \times r$ matrices with norm $\|\mathrm{x}\| = \max_\mu \big(\sum_\nu |\mathrm{x}_{\nu\mu}| \big)$ for $\mathrm{x} = (\mathrm{x}_{\nu\mu})$. Take $\mathrm{x} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}$. Since $\|\mathrm{x}\| = 1$, $L(P) \geq \|P(\mathrm{x})\|$ for any polynomial P. The matrix x is diagonalizable with eigenvalues $1, \rho, \dots, \rho^{r-1}$, where $\rho = e^{2\pi i/r}$. One can show that $\|P(\mathrm{x})\| = \frac{1}{r} \sum_{\mu=0}^{r-1} \left| \sum_{\nu=0}^{r-1} P(\rho^\nu)\rho^{\mu\nu} \right|$ for any polynomial P.

We may assume $|\mathfrak{a}_j| \geq 2$. Let $P(x) = q_M(x) p_{\mathfrak{g}}(x)(1 - x^k)^{-1}$. By choosing $r = 2k$ in the above discussion and since $q_M(\rho^\nu) p_{\mathfrak{g}}(\rho^\nu) = 2P(\rho^\nu)$ for $\rho = e^{\pi i/k}$ and all ν, it follows that $\|q_M(\mathrm{x}) p_{\mathfrak{g}}(\mathrm{x})\| = 2\|P(\mathrm{x})\|$. By induction it follows that $\|q_M(\mathrm{x}) p_{\mathfrak{g}}(\mathrm{x})\| = 2^{|\mathfrak{a}_k|-1} \|Q(\mathrm{x})\|$, where $Q(x) = q_M(x) p_{\mathfrak{g}}(x)(1 - x^k)^{1-|\mathfrak{a}_k|}$.

Since $q_M(\rho) \neq 0$ and $k > N/2$ by hypothesis, then $Q(\rho)$ is a non-zero eigenvalue of the integral matrix $Q(\mathrm{x})$. Therefore $\|Q(\mathrm{x})\| \geq 1$ and $\|q_M(\mathrm{x}) p_{\mathfrak{g}}(\mathrm{x})\| \geq 2^{|\mathfrak{a}_k|-1}$. Let $l \in \boldsymbol{N}$ be arbitrary. Since $\|q_M(\mathrm{x}) p_{\mathfrak{g}}(\mathrm{x})\| \geq \|(q_M(\mathrm{x}) p_{\mathfrak{g}}(\mathrm{x}))^l\|^{1/l} \geq (2^{l|\mathfrak{a}_k|-1})^{1/l} = 2^{|a_k|-1/l}$ holds for any l, we get that $\|q_M(\mathrm{x}) p_{\mathfrak{g}}(\mathrm{x})\| \geq 2^{|\mathfrak{a}_k|}$. □

Corollary 5.3. *Let $\mathfrak{g} = \oplus_{j=1}^N \mathfrak{a}_j$ be a graded Lie algebra and let $M = \oplus_{i=1}^m M_i$ be a graded $\mathfrak{g}$-module.*

If $m - 1 \leq \frac{N}{2}$, then

$$\big|\mathrm{H}_*(\mathfrak{g}, M)\big| \geq 2^R,$$

where $R = \max\{|\mathfrak{a}_j| : j > \frac{N}{2}\}$.

Proof. Take $k > \frac{N}{2}$, then $k > m-1$. Since $e^{\frac{t\pi i}{k}}$ has positive imaginary part for any t with $1 \leq t \leq m-1$, then $q_M(e^{\frac{\pi i}{k}}) \neq 0$. □

6. Applications

One of the open conjectures on the (co)-homology of nilpotent Lie algebras, known as *Toral Rank Conjecture* (TRC), claims that $2^{|\mathrm{center}(\mathfrak{g})|}$ is a lower bound for the total (co)-homology of $\mathfrak{g}$. In [7] it has been proved for 2-step Lie algebras and also it follows for the free nilpotent Lie algebras.

We prove here an analogue of the TRC for the homology of 2-step and free nilpotent Lie algebras with adjoint coefficients.

6.1. **Two-step nilpotent Lie algebras.** Let $\mathfrak{g}$ be a two-step Lie algebra, $\mathfrak{z}$ its center, and v any direct complement. Since $[v, v] \subseteq \mathfrak{z}$, $\mathfrak{g}$ admits the grading given by $\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ where $\mathfrak{a}_1 = v$ and $\mathfrak{a}_2 = \mathfrak{z}$.

Corollary 6.1. *Let $\mathfrak{g}$ be a 2-step Lie algebra and $\mathfrak{z}$ its center. Then*

$$\big|\mathrm{H}_*(\mathfrak{g}, \mathfrak{g})\big| \geq 2^{|\mathfrak{z}|}.$$

Proof. Immediate from Corollary 5.3 □

6.2. **Free nilpotent Lie algebras.** Let $r \geq 2$ and let $\mathcal{L}(r) = \oplus_{n \geq 1} H(n)$ be the free Lie algebra of rank r, where $H(n)$ is the space of n-brackets. The class N free nilpotent Lie algebra of rank r is

$$\mathcal{L}(N, r) = \frac{\mathcal{L}(r)}{\oplus_{n > N} H(n)};$$

the center $\mathfrak{z}$ of $\mathcal{L}(N, r)$ is $H(N)$.

The dimension $f_r(n) = |H(n)|$ can be computed by the formula

$$f_r(n) = \frac{1}{n} \sum_{d|n} \mu(d) r^{\frac{n}{d}},$$

where μ is the Möbius function.

Proposition 6.2. *Let $r \geq 3$ and let $f_r(n)$ be the dimension of the subspace of n-brackets of the free Lie algebra of rank r. Set $A_r(n-1) = \sum_{i=1}^{n-1} f_r(i)$. Then, $f_r(n) > A_r(n-1)$, except for $f_3(2) = A_3(1) = 3$.*

Proof. For small r's and small n's one can compute, with the help of a computer, $f_r(n)$ and verify by inspection that $f_r(n) > A_r(n-1)$. For this purpose we include the following table of values of $f_r(n)$.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$r=3$	3	3	8	18	48	116
$r=4$	4	6	20	60	204	670
$r=5$	5	10	40	150	624	2580
$r=6$	6	15	70	315	1554	7735
$r=7$	7	21	112	588	3360	19544
$r=8$	8	28	168	1008	6552	43596
$r=9$	9	36	240	1620	11808	88440
$r=10$	10	45	330	2475	19998	166485
$r=11$	11	55	440	3630	32208	295020
$r=12$	12	66	572	5148	49764	497354

For the general case we prove something stronger; in fact it holds that $f_r(n) > 2f_r(n-1)$, with a few exceptions, all contained in the previous table.

Let $\alpha = f_r(n) - 2f_r(n-1) = \frac{1}{n}\sum_{d|n} \mu(d) r^{\frac{n}{d}} - \frac{2}{n-1}\sum_{e|n-1} \mu(e) r^{\frac{n-1}{e}}$. Since any $m \in \mathbf{N}$ has at most m divisors, then $f_r(n)$ as at most m summands. The largest is $\frac{1}{n} r^n$ and all the others are $\geq -\frac{1}{n} r^{\frac{n}{2}}$. Therefore,

$$\begin{aligned} \alpha &> \tfrac{1}{n} r^n - n\tfrac{1}{n} r^{\frac{n}{2}} - \tfrac{2}{n-1} r^{n-1} - (n-1)\tfrac{2}{n-1} r^{\frac{n-1}{2}} \geq \\ &\geq \tfrac{1}{n} r^n - r^{\frac{n}{2}} - \tfrac{2}{n-1} r^{n-1} - 2r^{\frac{n}{2}} = \tfrac{1}{n} r^n - \tfrac{2}{n-1} r^{n-1} - 3r^{\frac{n}{2}}. \end{aligned}$$

Hence, it is enough to show that $r^{n-1}\big(r - 3nr^{(\frac{n}{2}-n+1)} - \frac{2n}{n-1}\big) > 0$, which is equivalent to

$$r - \frac{3n}{r^{\frac{n}{2}-1}} - \frac{2n}{n-1} > 0. \tag{5}$$

If $n \geq 4$, then $\frac{2n}{n-1} < 3$ and (5) will follow if $\frac{3n}{r^{\frac{n}{2}-1}} < 1$. One can check that this is true for any $r \geq 13$ (and $n \geq 4$). On the other hand, it also holds true for any $r \geq 4$ if $n \geq 7$.

Now it remains to consider the case $r = 3$. In this case the inequality in (5) becomes

$$3 - \frac{3n}{3^{\frac{n}{2}-1}} - \frac{2n}{n-1} > 0. \tag{6}$$

If $n \geq 6$, then $\frac{2n}{n-1} < \frac{5}{2}$ and (6) will follow if $\frac{3n}{3^{\frac{n}{2}-1}} < \frac{1}{2}$. This holds true for $n \geq 10$. By observing the table and noticing that $h_3(7) = 312$, $h_3(8) = 810$ and $h_3(9) = 2184$ the proposition is proved. □

Corollary 6.3. *Let $r \geq 3$ and let $\mathfrak{g} = \mathcal{L}(N,r)$. Then $\left|\mathrm{H}_*(\mathfrak{g},\mathfrak{g})\right| \geq 2^{|\mathfrak{z}|}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$.*

Proof. From Proposition 6.2 it follows that q_M, where $M = \mathfrak{g}$, has no roots of modulo 1. The corollary now follows from Theorem 5.2. □

Furthermore, since $\mathcal{L}(N,r)$ acts on $\mathcal{L}(r)$, by $H(j).\mathcal{L}(r) = [H(j), \mathcal{L}(r)]$, preserving the ideal $I_K = \oplus_{n>K} H(n)$, then $\mathcal{L}(N,r)$ acts on $M = \mathcal{L}(K,r)$.

Corollary 6.4. *Let $r \geq 3$, $\mathfrak{g} = \mathcal{L}(N,r)$ and $M = \mathcal{L}(K,r)$. Then $\left|\mathrm{H}_*(\mathfrak{g},M)\right| \geq 2^{|\mathfrak{z}|}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$.*

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